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Flow Research Note No. 178

Numerical Solution of Transonic Full Stream Function Equations in Conservation Form



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Jerry , Your comments are appreciated.
I am still working on the conservation results .

mohamed

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By

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October 1979

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Numerical Solution of Transonic Full Stream
Function Equations in Conservation Form^{*}

Abstract

The stream function equation in conservation form is solved iteratively based on the artificial compressibility method. The density is not a unique function of the mass flux. To avoid the ambiguity near the sonic line, the density is updated in terms of the velocity, which is obtained through a simple integration of a first order equation step-by-step in the flow field. Iteration algorithms and finite difference approximations are discussed, and numerical results of both conservative and nonconservative calculations are presented.

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1. Introduction

The stream function formulation is suitable for design problems and internal flow as well as rotational flow calculations. The stream lines and potential lines (in the case of irrotational flows) provide an orthogonal grid with a natural body fitted coordinate.

There are, however, some difficulties in solving the stream function equation: (1) the density is not a unique function of the mass flux, and (2) stream function exists only for two-dimensional and axisymmetric flows. Extension to three-dimensional flows in terms of more than one stream function is not simple to implement. Due to these stumbling blocks, the stream function has not been recently used.

In 1944 Emmons (Reference 1) solved the stream function equation by hand relaxation where the shock was fitted. In 1968 Sells (Reference 2) calculated only subcritical flows using line relaxations. Here transonic flows are calculated where shocks are captured, based on the artificial compressibility method (Reference 3), and the density is updated in terms of the velocity without ambiguity.

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2. Stream Function Formulation

For irrotational flows, the governing equations are:

$$(\rho u)_x + (\rho v)_y = 0 \quad (1)$$

$$u_y - v_x = 0 \quad (2)$$

$$\rho = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 (u^2 + v^2 - 1) \right]^{\frac{1}{\gamma - 1}} . \quad (3)$$

A potential function exists and it is governed by:

$$(\rho \phi_x)_x + (\rho \phi_y)_y = 0 \quad (4)$$

$$\rho = (M_\infty^2 a^2)^{\frac{1}{\gamma - 1}} = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 (\phi_x^2 + \phi_y^2 - 1) \right]^{\frac{1}{\gamma - 1}} . \quad (5)$$

In terms of natural coordinates s, n Equation (4) becomes:

$$(1 - M^2) \phi_{ss} + \phi_{nn} = 0 . \quad (6)$$

Similarly, a stream function exists such that:

$$\frac{-\psi_x}{\rho} = \phi_y = v , \quad \frac{\psi_y}{\rho} = \phi_x = u \quad (7)$$

$$\left(\frac{-\psi_s}{\rho} = \phi_n , \quad \frac{\psi_n}{\rho} = \phi_s \right) \quad (8)$$

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Hence

$$\left(\frac{\psi_x}{\rho}\right)_x + \left(\frac{\psi_y}{\rho}\right)_y = 0 \quad (9)$$

$$\rho = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{\psi_x^2 + \psi_y^2}{\rho^2} - 1 \right) \right]^{\frac{1}{\gamma - 1}} \quad (10)$$

or
$$(1 - M^2)\psi_{ss} + \psi_{nn} = 0 \quad . \quad (11)$$

While the density (the speed of sound) is uniquely determined in terms of the speed $q^2 = u^2 + v^2$, from Equation (3), Equation (10) indicates that there are two values of the density for certain mass flux (Sells, Reference 2, obtained rational approximations for the subsonic and supersonic branches).

We notice also that Equation (11) is a nonlinear mixed-type equation similar to Equation (6).

3. Artificial Viscosity

It is well known (Reference 3) that Equation (6) can be solved by adding an artificial viscosity term either explicitly or indirectly (through upwind differencing), namely,

$$(1 - M^2)\phi_{ss} + \phi_{nn} = -\epsilon\phi_{sss} \quad (12)$$

The corresponding stream function equation is

$$(1 - M^2)\psi_{ss} + \psi_{nn} = -\epsilon\psi_{sss} \quad (13)$$

Equation (13) is indeed equivalent to Equation (12) with the augmented artificial viscosity term.

Therefore, upwind differencing of Equation (11) in exactly the same way that Equation (6) is solved will yield nonconservative calculations. In the first case, however, the flow is irrotational and due to the nonconservative differencing, a proper mass balance across the shock is not achieved and the error can be represented as a source distribution at the shock. On the other hand, the mass is conserved automatically in the stream function formulation and due to the nonconservative differencing, a vorticity distribution at the shock is produced.

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4. Shock Jump Conditions

Equation (1) admits a discontinuous solution such that

$$[[\rho u]] - \left(\frac{dx}{dy} \right)_s [[\rho v]] = 0 \quad (14)$$

$$\left(\frac{dX}{dy} \right)_s [[u]] + [[v]] = 0 \quad (15)$$

In terms of the velocity potential, Equations (14 and 15) read

$$[[\rho \phi_x]] - \left(\frac{dX}{dy} \right)_s [[\rho \phi_y]] = 0 \quad (16)$$

$$[[\phi]] = 0 \quad (17)$$

and in terms of the stream function, they are:

$$[[\psi]] = 0 \quad (18)$$

$$\left(\frac{dX}{dy} \right)_s \left[\left[\frac{\psi_y}{\rho} \right] \right] - \left[\left[\frac{\psi_x}{\rho} \right] \right] = 0 \quad (19)$$

Either the shock is fitted according to the above shock jump relations, or it is captured through a proper conservative finite differencing, such that in the limit of small mesh, the shock jump conditioning are satisfied.

-6-

5. Artificial Compressibility

In Reference (3), the full potential equation in conservation form is solved. An artificial viscosity is added through modifying the density, namely,

$$(\tilde{\rho}\phi_x)_x + (\tilde{\rho}\phi_y)_y = 0 \quad (20)$$

where
$$\tilde{\rho} = \rho - \mu \rho_s \Delta_s \quad (21)$$

$$\rho_s \Delta_s \cong \frac{u}{q} \rho_x \Delta_x + \frac{v}{q} \rho_y \Delta_y \quad (22)$$

Centered differences are used everywhere in Equation (20). Only ρ_s is evaluated using upwind differencing.

Similarly, the artificial viscosity can be added to the stream function equation as follows:

$$u = \frac{\psi_y}{\rho} + \mu \frac{u}{\rho} \rho_x \Delta_x \quad (23)$$

$$v = \frac{-\psi_x}{\rho} - \mu \frac{v}{\rho} \rho_y \Delta_y \quad (24)$$

Hence,
$$\left(\frac{\psi_x}{\tilde{\rho}}\right)_x + \left(\frac{\psi_y}{\tilde{\rho}}\right)_y = 0 \quad (25)$$

or
$$u = \psi_y / \tilde{\rho} \quad (26)$$

$$v = -\psi_x / \tilde{\rho} \quad (27)$$

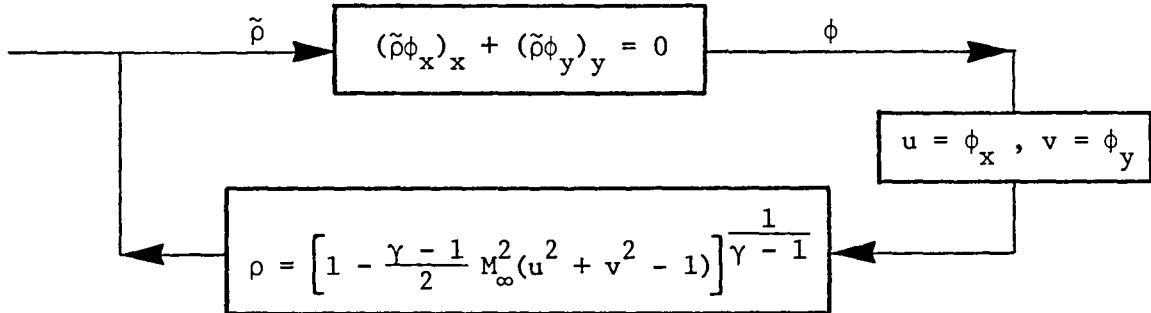
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and
$$\left(\frac{\psi_x}{\tilde{\rho}} \right)_x + \left(\frac{\psi_y}{\tilde{\rho}} \right)_y = 0 \quad (28)$$

-8-

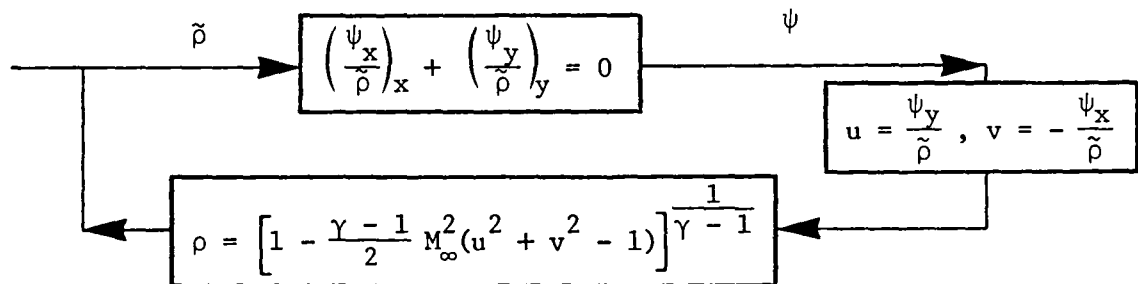
6. Iteration Algorithms

In the potential calculations, given the density from previous iterations, the velocity potential is calculated from Equation (20). Then the density is updated using Equation (5). The iteration loop is shown in Sketch 1.



Sketch 1

A similar iteration loop is implemented for stream function calculations as shown in Sketch 2. The algorithm converges only for subcritical flows.



Sketch 2

The iteration algorithm described in Sketch 2 is based on the following procedure:

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$$\rho_{\text{new}} = \left[1 - \frac{\gamma - 1}{2} M_{\infty}^2 \left(\frac{\psi_x^2 + \psi_y^2}{\rho_{\text{old}}^2} - 1 \right) \right] \frac{1}{\gamma - 1} \quad (29)$$

Notice that ρ_{new} decreases if the mass flux $(\psi_x^2 + \psi_y^2)$ increases. This is true only for subsonic flows. Equation (10) can be rewritten in the form:

$$F(\rho) = \rho^{\gamma+1} - \left(1 + \frac{\gamma - 1}{2} M_{\infty}^2 \right) \rho^2 + \frac{\gamma - 1}{2} M_{\infty}^2 (\psi_x^2 + \psi_y^2) = 0 \quad (30)$$

A Newton iteration on the single nonlinear equation in ρ (Equation (30)) is given by:

$$J(\rho_{\text{old}}) \cdot (\rho_{\text{new}} - \rho_{\text{old}}) = -F(\rho_{\text{old}})$$

where $J(\rho) = (\gamma + 1)\rho^{\gamma} - 2\rho\left(1 + \frac{\gamma - 1}{2} M_{\infty}^2\right)$ (31)

Notice that the Jacobian is negative or positive depending on whether ρ_{old} is larger than or smaller than ρ^* (ρ at $a = a^*$). For pure subsonic or supersonic flows, Equation (31) may be used. But for mixed flows, perturbing the sonic conditions yields two possible solutions, and there is no obvious way to exclude anyone of them based on local considerations, unless the sonic line is fitted.

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7. Mixed Flow Calculations

To avoid the ambiguity in the density updating, a different approach is adopted to calculate the speed given the stream function without referring to the old density. Starting with Equation (2):

$$u_y - v_x = 0$$

$$\text{or} \quad u_y - \left(\frac{v}{u} \cdot u \right)_x = 0 \quad (32)$$

$$\text{where} \quad \frac{v}{u} = - \frac{\psi_x}{\psi_y}$$

$$\text{Hence} \quad u_y + \left(\frac{\psi_x}{\psi_y} \cdot u \right)_x = 0 \quad (33)$$

$$\text{or} \quad (\ln u)_y + \frac{\psi_x}{\psi_y} \cdot (\ln u)_x = - \left(\frac{\psi_x}{\psi_y} \right)_x \quad (34)$$

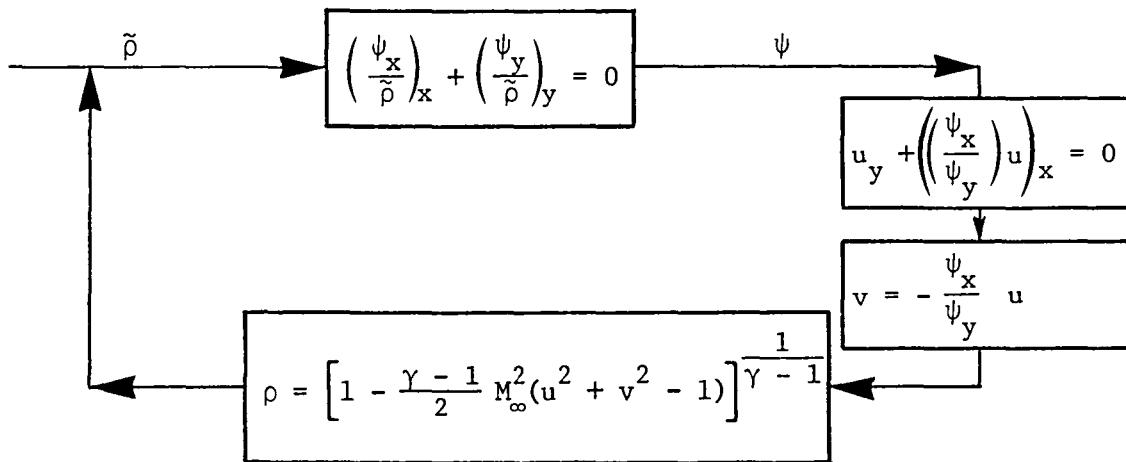
Knowing u on a curve crossing the characteristics of Equation (33), u can be calculated throughout the flow field step by step. The conditions along such a curve could be either pure subsonic or pure supersonic, where u can be obtained using a Newton iteration.

Once u is obtained, v is evaluated from Equation (35)

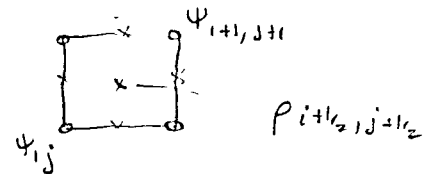
$$v = - \frac{\psi_x}{\psi_y} u \quad (35)$$

The iteration loop is shown in Sketch 3.

-11-



Sketch 3



$$\text{let } g = \frac{\psi_x}{\psi_y}$$

$$u_y + (gu)_x = 0$$

$$(u_y)_{i,j} = \frac{u_{i+1/2,j+1/2} - u_{i+1/2,j-1/2}}{2\Delta y} + \frac{u_{i-1/2,j+1/2} - u_{i-1/2,j-1/2}}{2\Delta y}$$

$$[(gu)_x]_{i,j} = \frac{(gu)_{i+1/2,j} - (gu)_{i-1/2,j}}{\Delta x}$$

$$(gu)_{i+1/2,j} = \frac{1}{2} (g_{i+1/2,j+1/2} u_{i+1/2,j+1/2} + g_{i+1/2,j-1/2} u_{i+1/2,j-1/2})$$

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8. Relaxation Method

Horizontal line over relaxation (with additional ϕ_{xt} term, Reference 3) of the partial differential equation for ψ , given $\tilde{\rho}$, is found to be fast. To explain this, consider the transonic small disturbance system of equations:

$$v_y = (u - K)u_x - \epsilon w_x$$

$$u_y = v_x$$

where dots
3rd eqn. comes from

$$\alpha w_y = -u_x + w \tag{36}$$

The characteristics of system (36) is given by

$$is \lambda = \frac{dx}{dy} \text{ or } \frac{dy}{dx} ?$$

$$\lambda^3 - (u - k)\lambda - \epsilon/\alpha = 0 \tag{37}$$

In the limit $\frac{\alpha}{\epsilon} \rightarrow 0$, the system (36) is parabolic with $y = \text{constant}$ as a triple characteristic. Similarly, the streamlines are the characteristics of Equation (12).

Moreover, the iterative matrix of the horizontal over relaxation procedure seems to have a dominant eigenvalue allowing for even more acceleration.

9. Numerical Results

Transonic flows over NACA 0012 airfoil have been calculated based on the full potential equation as well as the stream function equation. In these calculations, linearized boundary conditions are used. The stream function equation is solved using the iteration procedure described in Sketch 3. Both conservative and nonconservative results are presented. The iteration history of both vertical and horizontal line relaxations are plotted. Results for different magnitudes of the artificial viscosity in the artificial density formulation are also shown.

Finally, we would like to mention that rotational flows can be easily calculated by modifying Equation (2) where the righthand side becomes the vorticity $\omega(\psi)$. The density formula is also slightly modified by $e^{-s(\psi)}$ where $s(\psi)$ is the entropy. Also, other iterative procedures (i.e., ADI, Fast Solver, MAD, etc.) are, of course, applicable.

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Appendix Transonic Small Disturbance Equation

Chin and Rizzetta (Reference 4) solved the small disturbance equation:

$$(1 - M^2)\psi_{xx} + \psi_{yy} = 0 \quad (A-1)$$

where

$$1 - M^2 \cong 1 - M_\infty^2 - (\gamma + 1)M_\infty^2 u$$

Following Emmons (Reference 1), the small perturbation approximation of Equation (23), page 19,

$$\left(\frac{q}{q_1} = c(\xi) e^{-\int \frac{\psi_{\xi\xi}}{\psi_{\eta}} d\eta} \right)$$

is,

$$u_2 - u_1 = - \int_{y_1}^{y_2} \psi_{xx} dy \quad (A-2)$$

Equation (A-2) is consistent with the approximation

$$\phi_y = - \psi_x$$

hence

$$\phi_{yx} = - \psi_{xx}$$

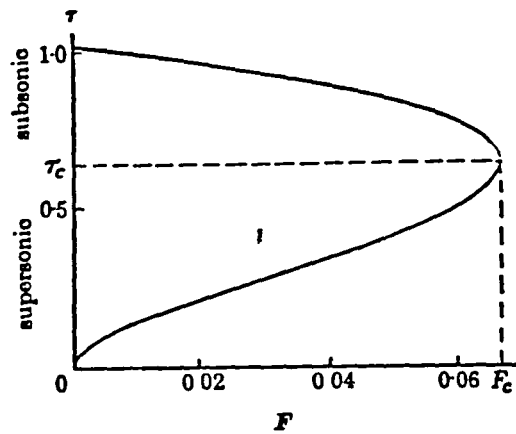
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or
$$\phi_{xy} = - \psi_{xx} \quad (A-3)$$

Chin's calculations are nonconservative, not only because of Equation (A-1) but also because of Equation (A-2).

References

1. Emmons, H. W. (1944) "The Numerical Solution of Compressible Fluid Flow Problems," NACA TN 932.
2. Sells, C. C. L. (1968) "Plane Subcritical Flow Past a Lifting Airfoil," Proc. Roy. Soc. 308A, p. 377-401, London.
3. Hafez, M. M., South, C. J., Murman, E. M. (1979) "Artificial Compressibility Methods for Numerical Solutions of Transonic Full Potential Equation," AIAA Journal 17, No. 8, p. 838-844, August.
4. Chin, W. C. and Rizzetta, D. P. "Airfoil Design in Subcritical and Supercritical Flows," to be published in Journal of Applied Mechanics.



$$\tau = \frac{0.6339395 + 1.421451 Q + 0.5431537 Q^2 - 0.1998823 Q^3}{1 + 1.242281 Q - 0.1511548 Q^2 - 0.0553184 Q^3}$$

$$\tau = \frac{0.6339388 - 0.8257193 Q - 0.8718085 Q^2 + 1.031123 Q^3}{1 - 0.3022746 Q - 1.449373 Q^2 + 0.3119677 Q^3}$$

$$\rho_s^{\gamma-1} = 1 + \frac{1}{2} M^2 (\gamma - 1)$$

$$a_s^2 = \rho_s^{\gamma-1} / M^2$$

$$\frac{f^2}{\rho_s^2 a_s^2} = \frac{2}{\gamma-1} \left[1 - \left(\frac{\rho}{\rho_s} \right)^{\gamma-1} \right] \left(\frac{\rho}{\rho_s} \right)^2$$

$$\rho / \rho_s = \tau; \quad \frac{1}{2} (\gamma - 1) f^2 / \rho_s^2 a_s^2 = F$$

$$F = \tau^2 (1 - \tau^{\gamma-1})$$

$$F_c = \left(\frac{2}{\gamma+1} \right)^{2(\gamma-1)} \left(\frac{\gamma-1}{\gamma+1} \right)$$

$$\tau_c = \left(\frac{2}{\gamma+1} \right)^{1/(\gamma-1)}$$

$$Q^2 = (F_c - F) \frac{2}{(\gamma^2 - 1) \tau_c^{\gamma+1}}$$

$$\gamma = 1.4,$$

$$Q = 1.44 [3(F_c - F)]^{1/2}$$

Figure 1. Density - Mass Flow Relationship
(Rational Approximation Formula After
Sells, Reference 2).

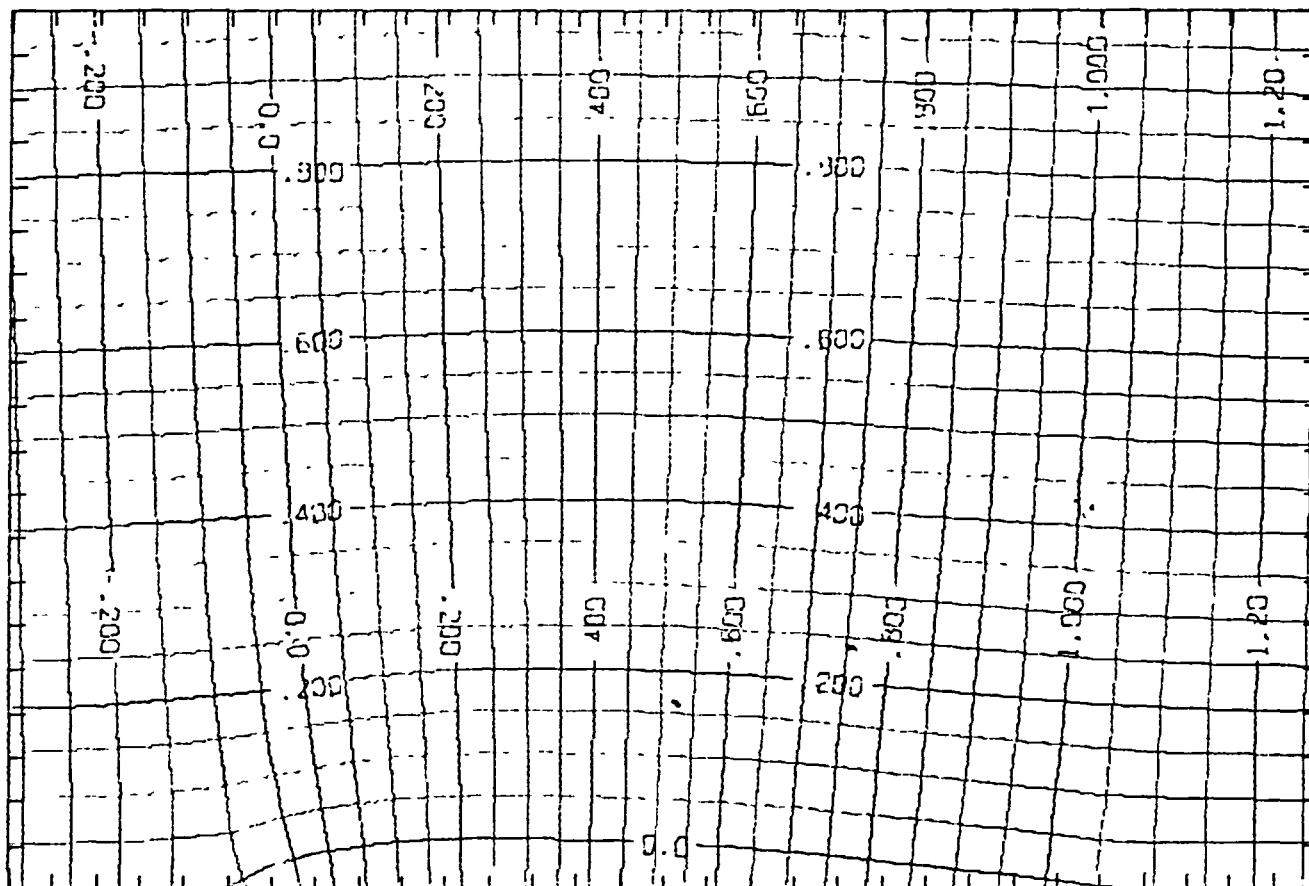
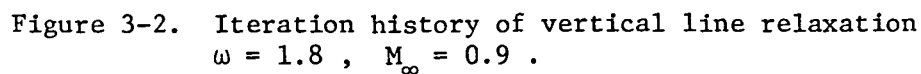


Figure 2. Potential and Stream Lines For Flows Around
NACA 0012, $M_{\infty} = 0.85$.

ITER	NAME5	1	2	10**(-5)	10**(-4)	10**(-3)	10**(-2)	10**(-1)	10*
1	1.33.95v22	25	2	5	4	3	2	1	0
2	12.040536	34	2	5	4	3	2	1	0
3	6.495335	35	2	5	4	3	2	1	0
4	1.434327	38	2	5	4	3	2	1	0
5	2.202015	27	7	5	4	3	2	1	0
6	1.675746	27	8	5	4	3	2	1	0
7	1.327726	27	9	5	4	3	2	1	0
8	1.002230	27	10	5	4	3	2	1	0
9	.901316	28	11	5	4	3	2	1	0
10	.772433	27	11	5	4	3	2	1	0
11	.668430	27	10	5	4	3	2	1	0
12	.623274	26	10	5	4	3	2	1	0
13	.560444	26	11	5	4	3	2	1	0
14	.514813	26	12	5	4	3	2	1	0
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16	.430247	26	13	5	4	3	2	1	0
17	.395020	26	13	5	4	3	2	1	0
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19	.343057	26	14	5	4	3	2	1	0
20	.320114	26	14	5	4	3	2	1	0
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22	.281345	26	15	5	4	3	2	1	0
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24	.249531	25	15	5	4	3	2	1	0
25	.236088	25	16	5	4	3	2	1	0
26	.224427	25	16	5	4	3	2	1	0
27	.212825	25	17	5	4	3	2	1	0
28	.202605	25	17	5	4	3	2	1	0
29	.192445	25	17	5	4	3	2	1	0
30	.183491	25	17	5	4	3	2	1	0
31	.175662	25	18	5	4	3	2	1	0
32	.167865	25	18	5	4	3	2	1	0
33	.160371	25	18	5	4	3	2	1	0
34	.153431	24	18	5	4	3	2	1	0
35	.146795	24	18	5	4	3	2	1	0
36	.140471	24	18	5	4	3	2	1	0
37	.133444	24	18	5	4	3	2	1	0
38	.127894	23	18	5	4	3	2	1	0
39	.122047	23	18	5	4	3	2	1	0
40	.116192	23	18	5	4	3	2	1	0
41	.110744	22	18	5	4	3	2	1	0
42	.105434	22	18	5	4	3	2	1	0
43	.100201	22	17	5	4	3	2	1	0
44	.095316	21	18	5	4	3	2	1	0
45	.090612	21	17	5	4	3	2	1	0
46	.086014	21	17	5	4	3	2	1	0
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50	.069497	19	17	5	4	3	2	1	0
51	.065987	19	17	5	4	3	2	1	0
52	.062595	18	17	5	4	3	2	1	0
53	.059270	19	17	5	4	3	2	1	0
54	.056152	17	17	5	4	3	2	1	0
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61	.038007	14	17	5	4	3	2	1	0
62	.035594	14	17	5	4	3	2	1	0
63	.033450	14	16	5	4	3	2	1	0
64	.030950	14	16	5	4	3	2	1	0
65	.030196	13	17	5	4	3	2	1	0
66	.028492	13	16	5	4	3	2	1	0
67	.026632	13	16	5	4	3	2	1	0
68	.025217	13	16	5	4	3	2	1	0
69	.023724	12	16	5	4	3	2	1	0
70	.022284	12	16	5	4	3	2	1	0
71	.020840	12	16	5	4	3	2	1	0
72	.019573	12	15	5	4	3	2	1	0
73	.018322	11	16	5	4	3	2	1	0
74	-.018130	30	9	5	4	3	2	1	0
75	-.017681	30	9	5	4	3	2	1	0
76	-.017623	30	8	5	4	3	2	1	0
77	-.017535	30	5	5	4	3	2	1	0
78	-.017407	30	5	5	4	3	2	1	0
79	-.017155	30	5	5	4	3	2	1	0
80	-.016808	30	5	5	4	3	2	1	0
81	-.016569	29	8	5	4	3	2	1	0
82	-.016305	29	8	5	4	3	2	1	0
83	-.015946	29	8	5	4	3	2	1	0
84	-.015631	29	8	5	4	3	2	1	0
85	-.015044	29	8	5	4	3	2	1	0
86	-.014470	29	9	5	4	3	2	1	0
87	-.013904	29	8	5	4	3	2	1	0
88	-.013256	29	9	5	4	3	2	1	0
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91	-.011842	28	18	5	4	3	2	1	0
92	-.011572	28	18	5	4	3	2	1	0
93	-.011195	28	18	5	4	3	2	1	0
94	-.010432	28	18	5	4	3	2	1	0
95	-.009176	27	11	5	4	3	1	1	0
96	-.008706	27	11	5	4	3	1	1	0
97	-.009391	27	11	5	4	3	1	1	0
98	-.008497	27	11	5	4	3	1	1	0
99	-.008600	27	11	5	4	3	1	1	0
100	-.008204	27	11	5	4	3	1	1	0
101	-.007854	26	12	5	4	3	1	1	0

Figure 3-1. Iteration history of vertical line relaxation
 $\omega = 1.8$, $M_{\infty} = 0.85$.



ITEM	NAME	I	J	10**(-5)	10**(-4)	10**(-3)	10**(-2)	10**(-1)	10**
1	1.3e-23750	24	2	5	4	3	2	1	0
2	1.1e-23750	24	2	5	4	3	2	1	0
3	0.5e-23750	24	2	5	4	3	2	1	0
4	0.2e-23750	24	2	5	4	3	2	1	0
5	2.2773e-5	20	7	5	4	3	2	1	0
6	1.7554e-5	20	6	5	4	3	2	1	0
7	1.0127e-1	20	6	5	4	3	2	1	0
8	1.1678e-8	20	10	5	4	3	2	1	0
9	0.9943e-1	20	11	5	4	3	2	1	0
10	0.8769e-8	20	9	5	4	3	2	1	0
11	0.7994e-13	20	10	5	4	3	2	1	0
12	0.7202e-5	20	11	5	4	3	2	1	0
13	0.6622e-7	27	11	5	4	3	2	1	0
14	0.6104e-3	20	12	5	4	3	2	1	0
15	0.5615e-2	20	13	5	4	3	2	1	0
16	0.5215e-8	20	14	5	4	3	2	1	0
17	0.4871e-1	20	15	5	4	3	2	1	0
18	0.4561e-3	20	15	5	4	3	2	1	0
19	0.4297e-7	20	14	5	4	3	2	1	0
20	0.4047e-3	20	14	5	4	3	2	1	0
21	0.3824e-9	20	15	5	4	3	2	1	0
22	0.3626e-4	20	15	5	4	3	2	1	0
23	0.3445e-5	20	15	5	4	3	2	1	0
24	0.3281e-5	20	16	5	4	3	2	1	0
25	0.3137e-5	20	16	5	4	3	2	1	0
26	0.2990e-5	20	16	5	4	3	2	1	0
27	0.2871e-5	20	17	5	4	3	2	1	0
28	0.2756e-7	20	17	5	4	3	2	1	0
29	0.2651e-5	20	18	5	4	3	2	1	0
30	0.2556e-4	20	18	5	4	3	2	1	0
31	0.2464e-2	20	18	5	4	3	2	1	0
32	0.2376e-4	20	18	5	4	3	2	1	0
33	0.2295e-3	20	19	5	4	3	2	1	0
34	0.2216e-3	20	19	5	4	3	2	1	0
35	0.2140e-1	20	19	5	4	3	2	1	0
36	0.2070e-2	20	19	5	4	3	2	1	0
37	0.1999e-1	20	19	5	4	3	2	1	0
38	0.1927e-8	20	19	5	4	3	2	1	0
39	0.1854e-7	20	19	5	4	3	2	1	0
40	0.1787e-7	27	19	5	4	3	2	1	0
41	0.1722e-2	27	19	5	4	3	2	1	0
42	0.1655e-7	27	19	5	4	3	2	1	0
43	0.1587e-7	27	18	5	4	3	2	1	0
44	0.1525e-1	20	18	5	4	3	2	1	0
45	0.1466e-4	20	18	5	4	3	2	1	0
46	0.1405e-8	20	18	5	4	3	2	1	0
47	0.1344e-5	20	18	5	4	3	2	1	0
48	0.1286e-3	25	18	5	4	3	2	1	0
49	0.1231e-3	25	18	5	4	3	2	1	0
50	0.1175e-5	25	18	5	4	3	2	1	0
51	0.1119e-4	25	18	5	4	3	2	1	0
52	0.1064e-5	24	18	5	4	3	2	1	0
53	0.1016e-7	24	18	5	4	3	2	1	0
54	0.0968e-4	24	18	5	4	3	2	1	0
55	0.0916e-4	23	18	5	4	3	2	1	0
56	0.0874e-7	23	18	5	4	3	2	1	0
57	0.0830e-9	23	18	5	4	3	2	1	0
58	0.0786e-6	23	18	5	4	3	2	1	0
59	0.0747e-1	22	18	5	4	3	2	1	0
60	0.0708e-2	22	18	5	4	3	2	1	0
61	0.0671e-3	22	18	5	4	3	2	1	0
62	0.0636e-0	21	18	5	4	3	2	1	0
63	0.0603e-2	21	18	5	4	3	2	1	0
64	0.0571e-1	20	19	5	4	3	2	1	0
65	0.0541e-4	20	19	5	4	3	2	1	0
66	0.0513e-6	19	19	5	4	3	2	1	0
67	0.0487e-1	19	19	5	4	3	2	1	0
68	0.0462e-1	18	19	5	4	3	2	1	0
69	0.0439e-3	17	20	5	4	3	2	1	0
70	0.0418e-1	17	20	5	4	3	2	1	0
71	0.0397e-9	16	20	5	4	3	2	1	0
72	0.0379e-1	16	20	5	4	3	2	1	0
73	0.0361e-2	16	20	5	4	3	2	1	0
74	0.0344e-4	15	20	5	4	3	2	1	0
75	0.0328e-8	15	20	5	4	3	2	1	0
76	0.0313e-0	15	20	5	4	3	2	1	0
77	0.0299e-2	14	20	5	4	3	2	1	0
78	0.0285e-2	14	20	5	4	3	2	1	0
79	0.0272e-9	14	20	5	4	3	2	1	0
80	0.0260e-6	13	20	5	4	3	2	1	0
81	0.0249e-8	13	20	5	4	3	2	1	0
82	0.0238e-1	13	20	5	4	3	2	1	0
83	0.0227e-3	13	20	5	4	3	2	1	0
84	0.0217e-2	12	20	5	4	3	2	1	0
85	0.0206e-4	12	20	5	4	3	2	1	0
86	0.0195e-7	12	20	5	4	3	2	1	0
87	0.0190e-3	12	20	5	4	3	2	1	0
88	0.0182e-1	11	20	5	4	3	2	1	0
89	0.0175e-1	11	20	5	4	3	2	1	0
90	0.0166e-3	11	20	5	4	3	2	1	0
91	0.0160e-3	11	20	5	4	3	2	1	0
92	0.0152e-2	10	20	5	4	3	2	1	0
93	0.0145e-2	10	20	5	4	3	2	1	0
94	0.0138e-5	10	20	5	4	3	2	1	0
95	0.0131e-6	10	20	5	4	3	2	1	0
96	0.0124e-3	10	20	5	4	3	2	1	0
97	0.0117e-2	10	20	5	4	3	2	1	0
98	0.0110e-1	10	20	5	4	3	2	1	0
99	0.0103e-7	10	20	5	4	3	2	1	0
100	0.0096e-1	10	20	5	4	3	2	1	0

Figure 3-3. Iteration history of vertical line relaxation
 $\omega = 1.8$, $M_{\infty} = 0.95$.

ITER	MAXRES	I	J	10**(-5)	10**(-4)	10**(-3)	10**(-2)	10**(-1)	
1	1.3482728	24	2	5	4	3	2	1	
2	11.966537	24	2	5	4	3	2	1	
3	6.563245	25	2	5	4	3	2	1	
4	1.621874	28	2	5	4	3	2	1	
5	2.311511	28	7	5	4	3	2	1	
6	1.766121	28	8	5	4	3	2	1	
7	1.437321	28	9	5	4	3	2	1	
8	1.198373	29	10	5	4	3	2	1	
9	1.017885	29	11	5	4	3	2	1	
10	991371	28	9	5	4	3	2	1	
11	.824477	28	10	5	4	3	2	1	
12	.752354	29	11	5	4	3	2	1	
13	.688374	29	12	5	4	3	2	1	
14	.633894	28	12	5	4	3	2	1	
15	.585725	29	13	5	4	3	2	1	
16	.543483	28	13	5	4	3	2	1	
17	.508954	28	13	5	4	3	2	1	
18	.479453	29	14	5	4	3	2	1	
19	.450764	29	14	5	4	3	2	1	
20	.426441	29	15	5	4	3	2	1	
21	.403955	29	15	5	4	3	2	1	
22	.381207	29	16	5	4	3	2	1	
23	.365445	29	16	5	4	3	2	1	
24	.348744	29	16	5	4	3	2	1	
25	.333414	29	17	5	4	3	2	1	
26	.319442	29	17	5	4	3	2	1	
27	.306646	29	17	5	4	3	2	1	
28	.294572	29	18	5	4	3	2	1	
29	.283774	29	18	5	4	3	2	1	
30	.273322	29	18	5	4	3	2	1	
31	.263444	29	19	5	4	3	2	1	
32	.254464	29	19	5	4	3	2	1	
33	.245294	29	20	5	4	3	2	1	
34	.237334	29	20	5	4	3	2	1	
35	.229474	29	20	5	4	3	2	1	
36	.221714	29	20	5	4	3	2	1	
37	.214551	29	20	5	4	3	2	1	
38	.207894	29	19	5	4	3	2	1	
39	.201130	29	19	5	4	3	2	1	
40	.193365	29	19	5	4	3	2	1	
41	.186507	27	20	5	4	3	2	1	
42	.180055	27	20	5	4	3	2	1	
43	.173502	28	19	5	4	3	2	1	
44	.166924	28	19	5	4	3	2	1	
45	.160542	28	20	5	4	3	2	1	
46	.155312	28	20	5	4	3	2	1	
47	.149440	28	20	5	4	3	2	1	
48	.143424	28	20	5	4	3	2	1	
49	.138002	25	20	5	4	3	2	1	
50	.132401	25	19	5	4	3	2	1	
51	.127564	25	19	5	4	3	2	1	
52	.123430	37	9	5	4	3	2	1	
53	.122445	37	10	5	4	3	2	1	
54	.119444	37	10	5	4	3	2	1	
55	.112351	37	11	5	4	3	2	1	
56	.110440	37	11	5	4	3	2	1	
57	.102716	37	11	5	4	3	2	1	
58	.098808	37	12	5	4	3	2	1	
59	.091564	37	12	5	4	3	2	1	
60	.085532	37	13	5	4	3	2	1	
61	.083757	36	8	5	4	3	2	1	
62	.082032	36	8	5	4	3	2	1	
63	.079751	36	9	5	4	3	2	1	
64	.076315	36	9	5	4	3	2	1	
65	.072503	36	10	5	4	3	2	1	
66	.067620	36	10	5	4	3	2	1	
67	.063744	35	9	5	4	3	2	1	
68	.060136	35	10	5	4	3	2	1	
69	.056468	35	10	5	4	3	2	1	
70	.051542	35	11	5	4	3	2	1	
71	.048324	19	21	5	4	3	2	1	
72	.046085	18	21	5	4	3	2	1	
73	.043697	18	21	5	4	3	2	1	
74	.041864	17	21	5	4	3	2	1	
75	.039990	17	21	5	4	3	2	1	
76	.038098	17	22	5	4	3	2	1	
77	.036451	16	22	5	4	3	2	1	
78	.034973	16	22	5	4	3	2	1	
79	.033442	16	22	5	4	3	2	1	
80	.031873	16	22	5	4	3	2	1	
81	.030573	15	22	5	4	3	2	1	
82	.029358	15	22	5	4	3	2	1	
83	.028047	15	22	5	4	3	2	1	
84	.026798	15	22	5	4	3	2	1	
85	.025627	14	22	5	4	3	2	1	
86	.024673	14	22	5	4	3	2	1	
87	.023673	14	22	5	4	3	2	1	
88	.022635	14	22	5	4	3	2	1	
89	.021570	14	22	5	4	3	2	1	
90	.020444	13	22	5	4	3	2	1	
91	.019311	13	22	5	4	3	2	1	
92	.018144	13	22	5	4	3	2	1	
93	.016934	13	22	5	4	3	2	1	
94	.015641	36	8	5	4	3	2	1	
95	.0147005	36	8	5	4	3	2	1	
96	.0137641	36	9	5	4	3	2	1	
97	.0126979	36	9	5	4	3	2	1	
98	.0116744	36	9	5	4	3	2	1	
99	.0106369	36	9	5	4	3	2	1	
100	.0106369	36	9	5	4	3	2	1	

Figure 3-4. Iteration history of vertical line relaxation

$$\omega = 1.8, \quad M_{\infty} = 0.98.$$

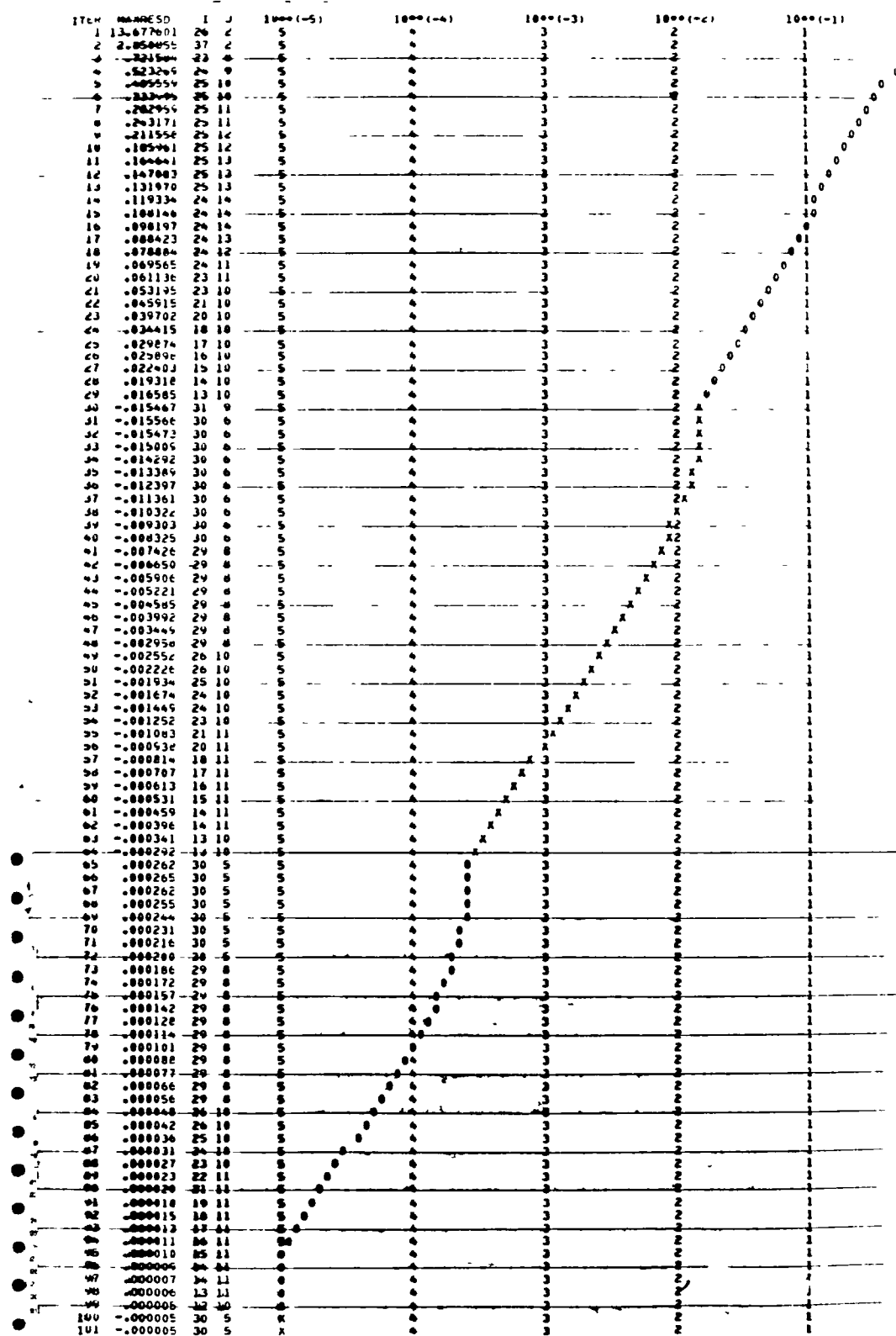


Figure 4-1. Iteration history of horizontal line relaxation
 $\omega = 1.8$, $M_{\infty} = 0.85$.

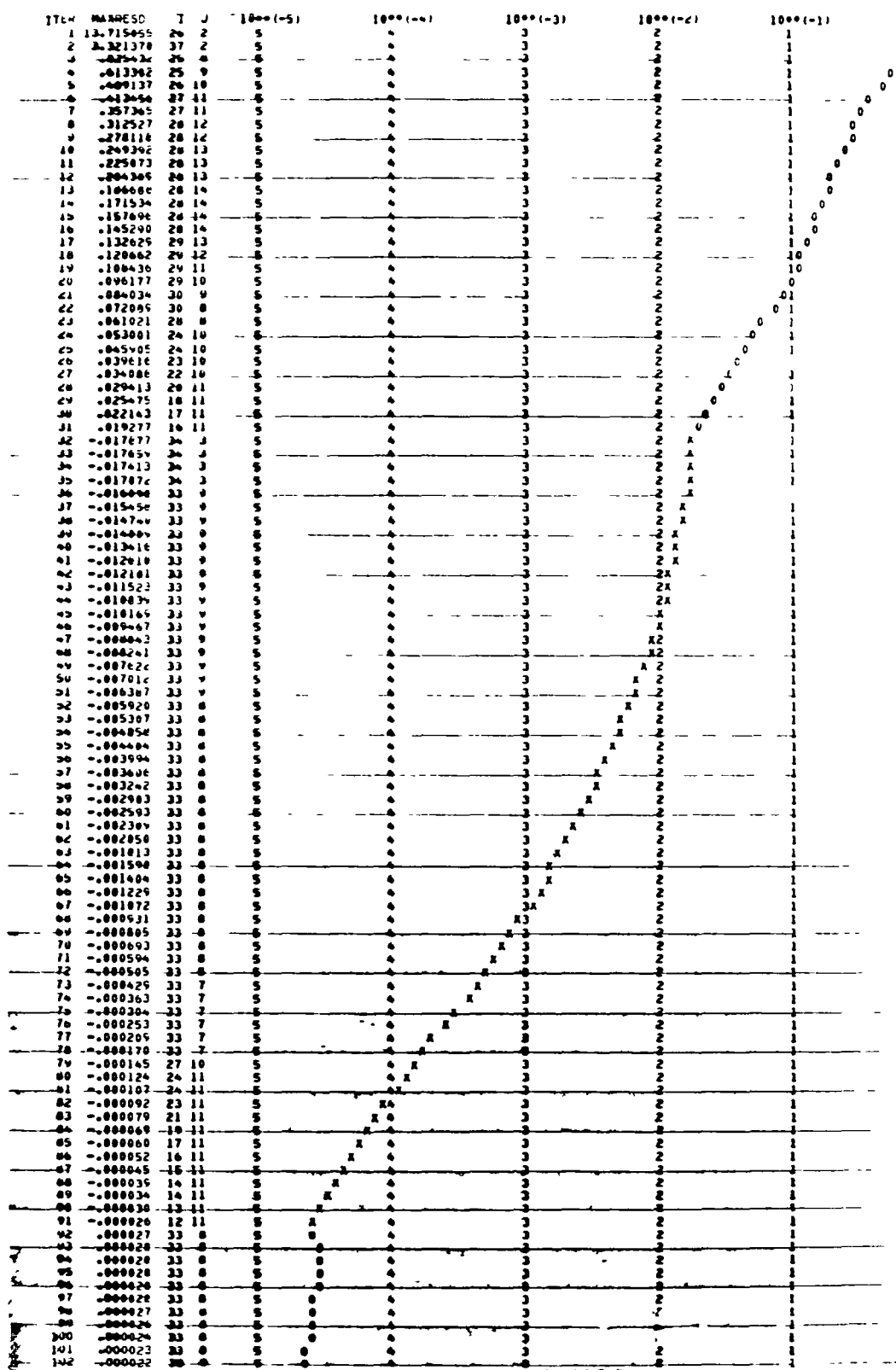


Figure 4-2. Iteration history of horizontal line relaxation
 $\omega = 1.8$, $M_{\infty} = 0.9$.

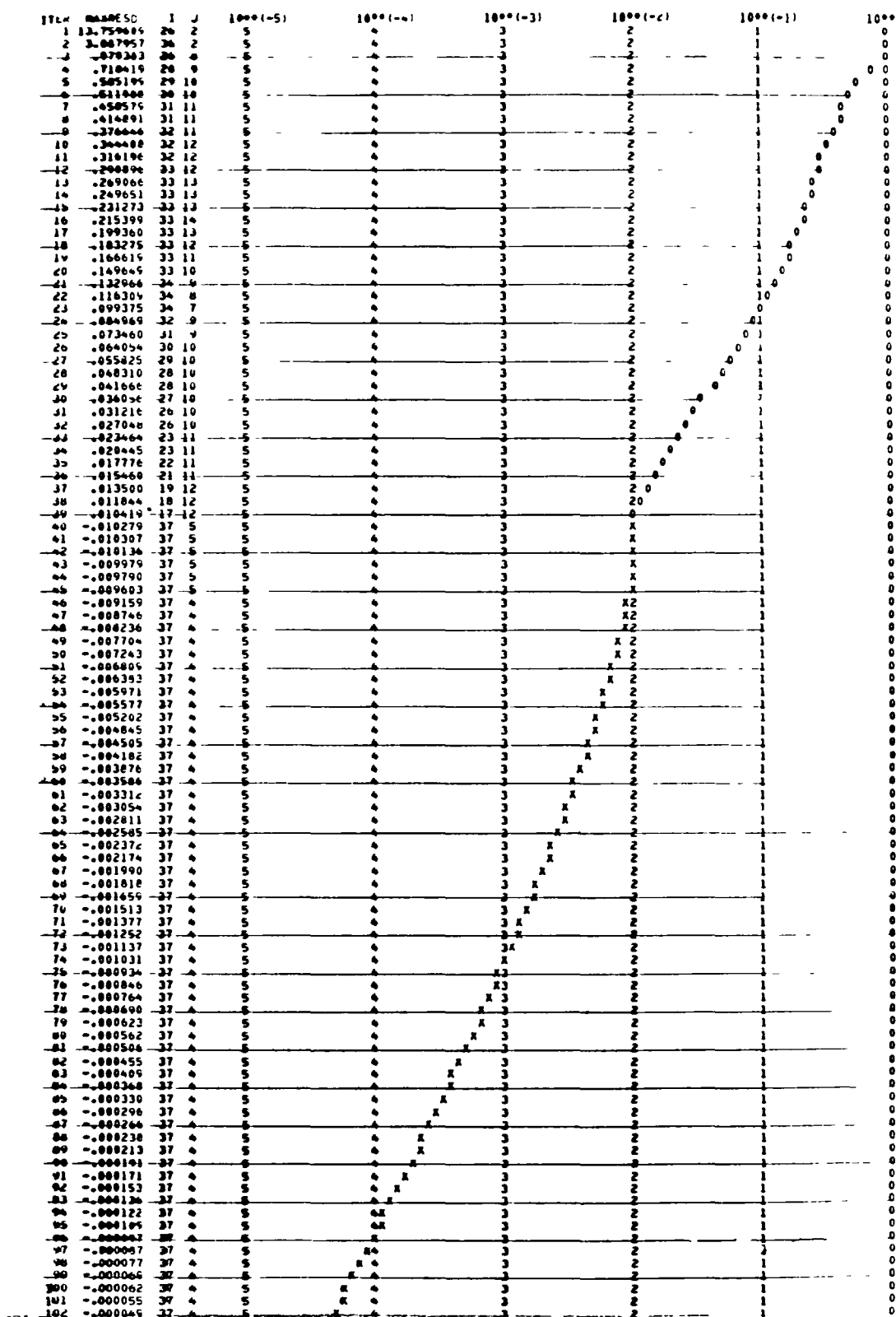


Figure 4-3. Iteration history of horizontal line relaxation
 $\omega = 1.8$, $M_{\infty} = 0.95$.

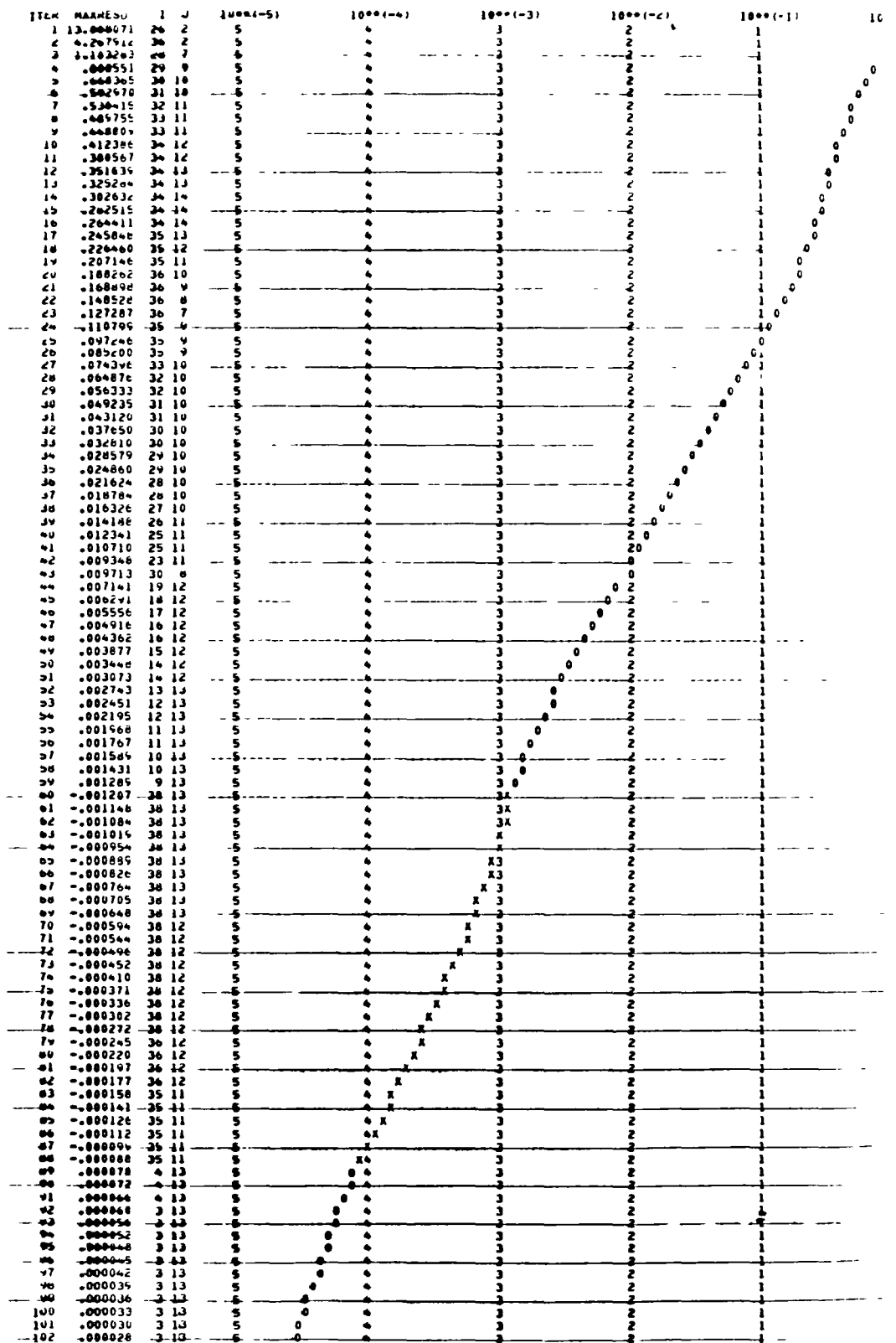


Figure 4-4. Iteration history of horizontal line relaxation
 $\omega = 1.8$, $M_{\infty} = 0.98$.

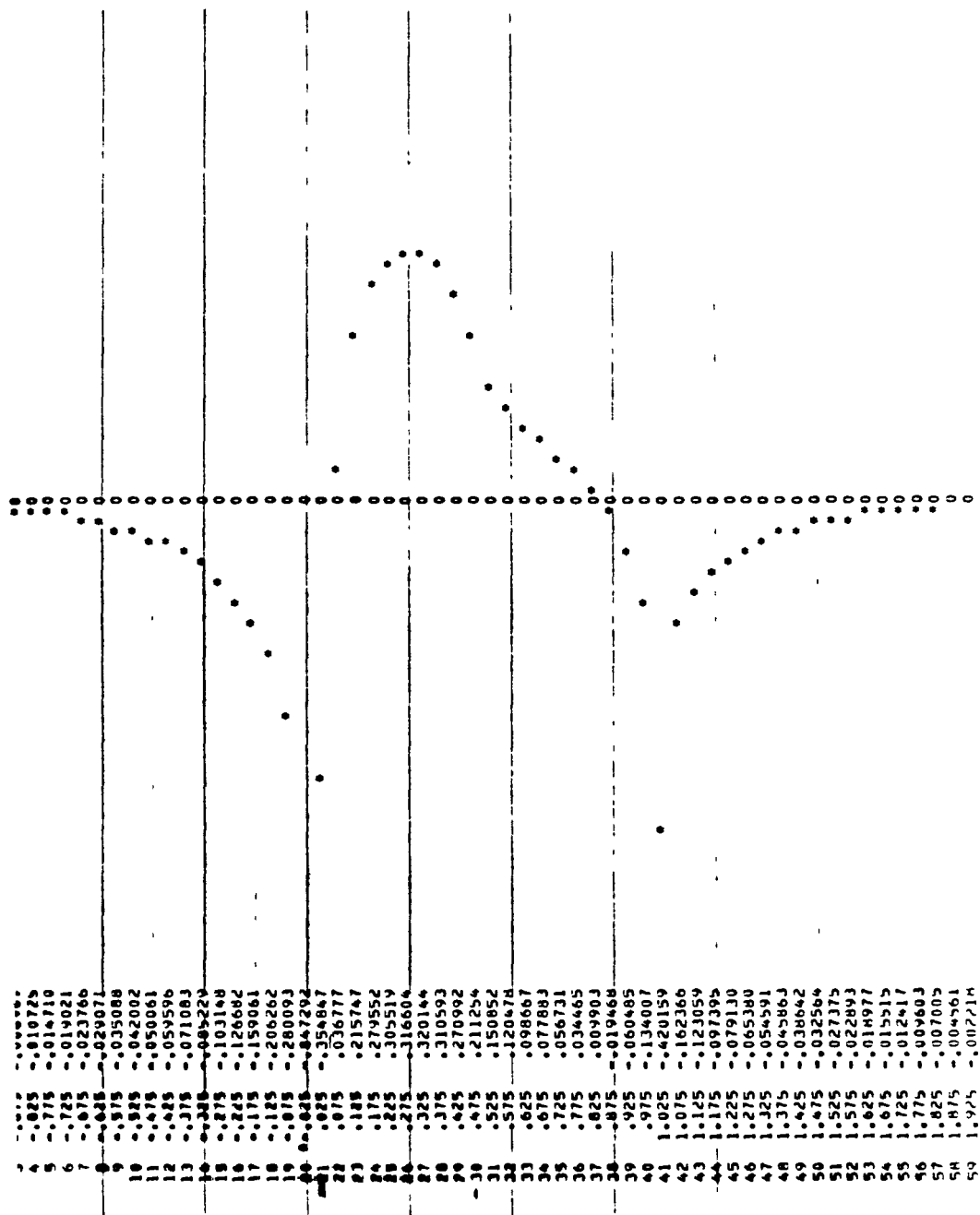


Figure 5-1. u versus x , $M_{\infty} = 0.85$, viscosity factor = 1.3.

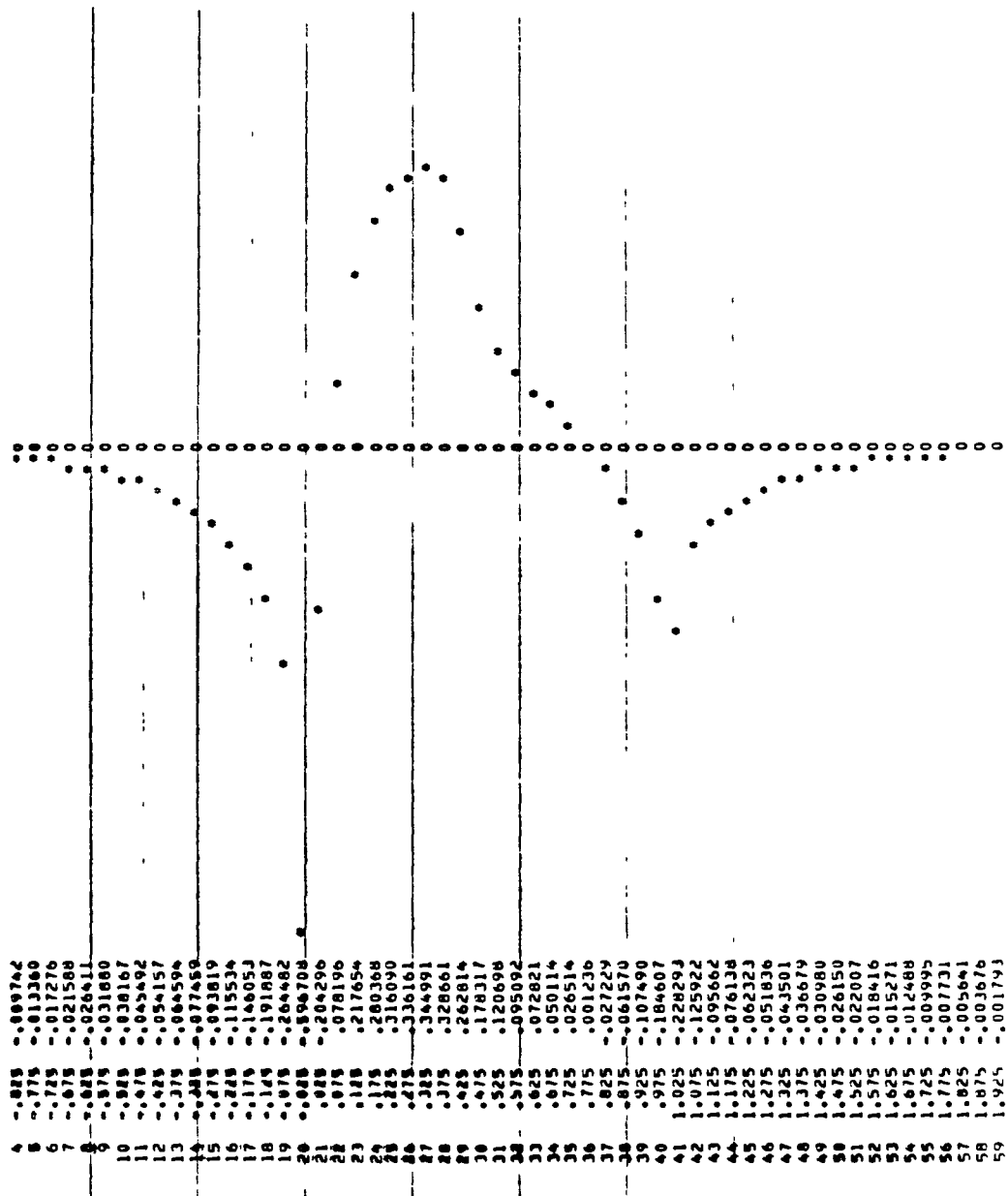


Figure 5-2. u versus x , $M_{\infty} = 0.85$, viscosity factor = 1.0.

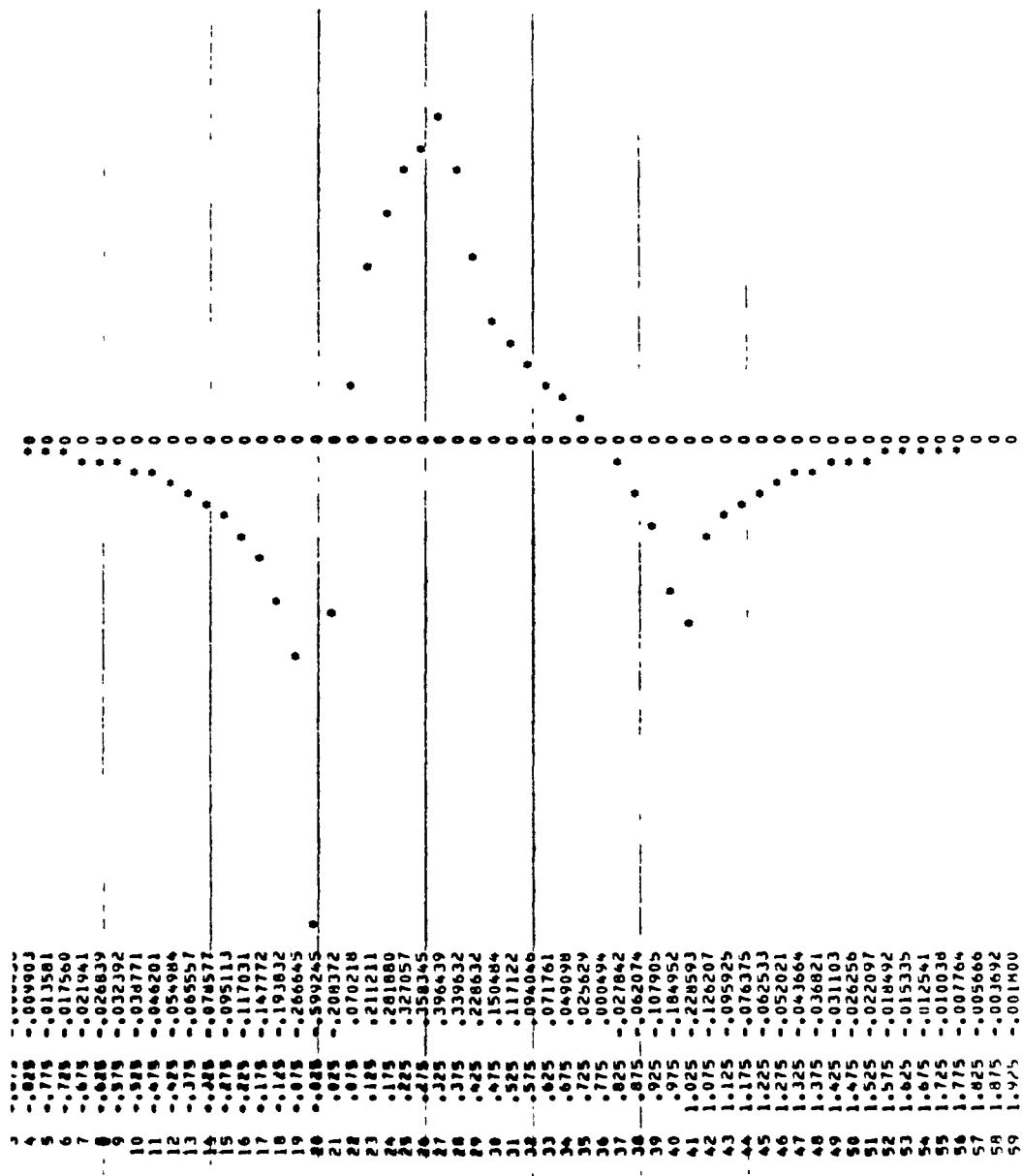


Figure 5-3. u versus x , $M_{\infty} = 0.85$, viscosity factor = 0.5.

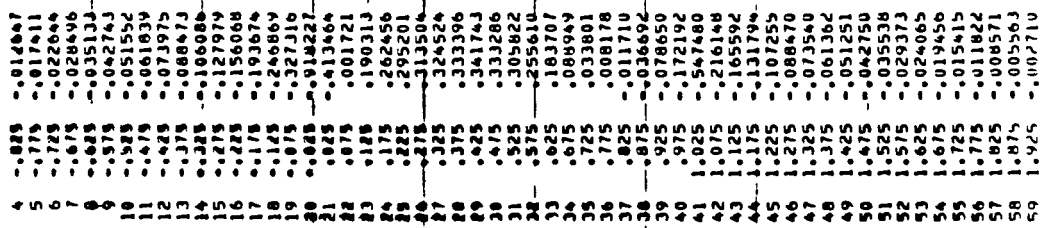


Figure 5-4. u versus x , $M_{\infty} = 0.9$.

4	-.025	-.013079
5	.775	-.019343
6	-.725	-.025570
7	-.075	-.032797
8	-.625	-.041207
9	-.975	-.051026
10	-.325	-.062492
11	-.675	-.075887
12	-.425	-.091573
13	-.375	-.110052
14	-.225	-.132064
15	-.275	-.158787
16	-.225	-.192140
17	-.175	-.235481
18	-.125	-.294922
19	-.075	-.381903
20	-.025	-.599484
21	.025	-.482769
22	.075	-.044876
23	.125	.157112
24	.175	.236185
25	.225	.272940
26	.275	.294409
27	.325	.307070
28	.375	.316668
29	.425	.324518
30	.475	.333535
31	.525	.342246
32	.575	.344406
33	.625	.334470
34	.675	.305822
35	.725	.256086
36	.775	.186313
37	.825	.081065
38	.875	-.043372
39	.925	-.132888
40	.975	-.287651
41	1.025	-.771952
42	1.075	-.322340
43	1.125	-.254176
44	1.175	-.206598
45	1.225	-.170778
46	1.275	-.142392
47	1.325	-.119098
48	1.375	-.099554
49	1.425	-.082938
50	1.475	-.068716
51	1.525	-.056516
52	1.575	-.046056
53	1.625	-.037104
54	1.675	-.029454
55	1.725	-.022911
56	1.775	-.017282
57	1.825	-.012373
58	1.875	-.007981
59	1.925	-.003904

Figure 5-5. u versus x , $M_\infty = 0.95$.

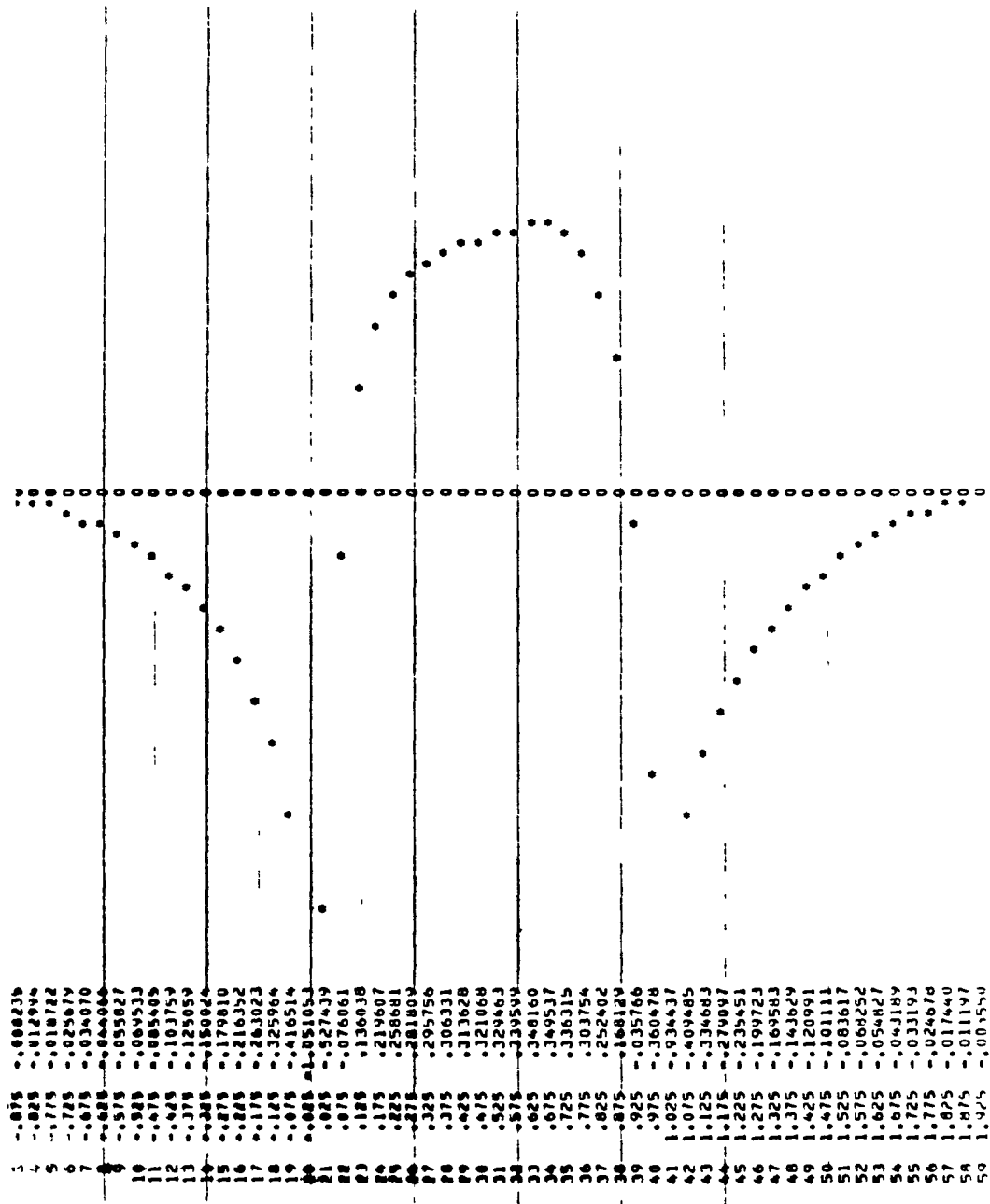


Figure 5-6. u versus x , $M_{\infty} = 0.98$.

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